SYDNEY GRAMMAR SCHOOL

2010 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 3rd August 2010

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist
- SGS booklets — 8 per boy
- Candidature — 89 boys

Examiner
PKH

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. **2**

(b) (i) Find the values of A and B such that $\dfrac{2x-1}{(x-1)^2} = \dfrac{A}{(x-1)^2} + \dfrac{B}{x-1}$. **2**

(ii) Hence find $\displaystyle\int \frac{2x-1}{(x-1)^2}\,dx$. **2**

(c) Find $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$. **2**

(d) Find $\displaystyle\int x\sec^2 x\,dx$. **3**

(e) Use the substitution $x = 2\tan\theta$ to find $\displaystyle\int_0^2 \frac{1}{\left(4+x^2\right)^2}\,dx$. **4**

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3+i$ and $w = 2-i$. Find in the form $x+iy$:

(i) wz **1**

(ii) $\overline{w}\,\overline{z}$ **1**

(iii) $\dfrac{w}{z}$ **1**

(b) (i) Write $1 - i\sqrt{3}$ in modulus-argument form. **2**

(ii) Find $(1-i\sqrt{3})^9$ in the form $a+ib$, where a and b are real. **2**

(c) Simplify $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$. **2**

(d) Sketch the region in the complex plane which simultaneously satisfies **3**

$|z-3-4i| \le 5$ and $\dfrac{3\pi}{4} \le \arg(z-3-4i) \le \pi$.

(e) (i) By letting $z = x+iy$, find the locus of points in the complex plane which satisfy **2**

$$\mathrm{Re}\left(z - \frac{1}{\overline{z}}\right) = 0.$$

(ii) Sketch the locus. **1**

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

(i) Find the eccentricity of the ellipse. **1**

(ii) Find the coordinates of the foci and the equations of the directrices. **2**

(iii) Sketch the ellipse, showing the foci and directrices. **1**

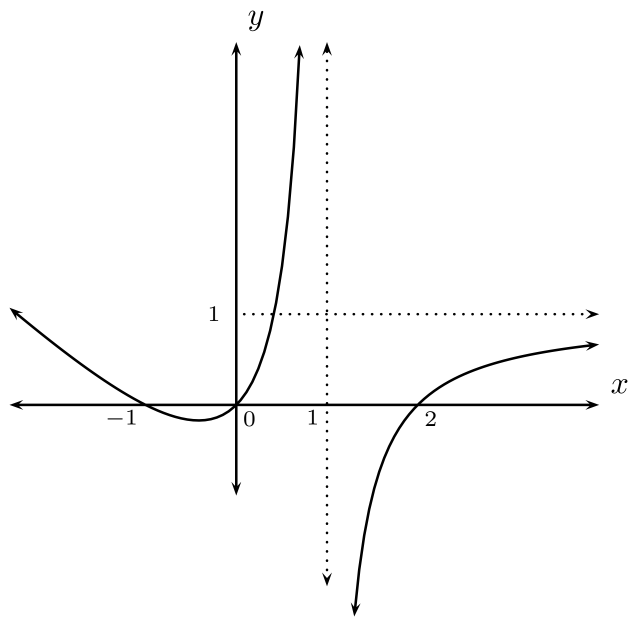
(b) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with eccentricity e. Let $P(x_1, y_1)$ be a point on the ellipse in the first quadrant.

(i) Show that the normal to the ellipse at P has gradient $\dfrac{a^2 y_1}{b^2 x_1}$. **1**

(ii) The normal at P cuts the x-axis at G. Show that the x-coordinate of G is $e^2 x_1$. **2**

(iii) If S is the focus $(ae, 0)$, show that $SG = e\,PS$. **2**

(c)



The diagram above shows the graph of $y = f(x)$, where the lines $x = 1$ and $y = 1$ are asymptotes. Draw separate one-third page sketches of the graphs of:

(i) $y = f(-x)$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

Exam continues overleaf . . .

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

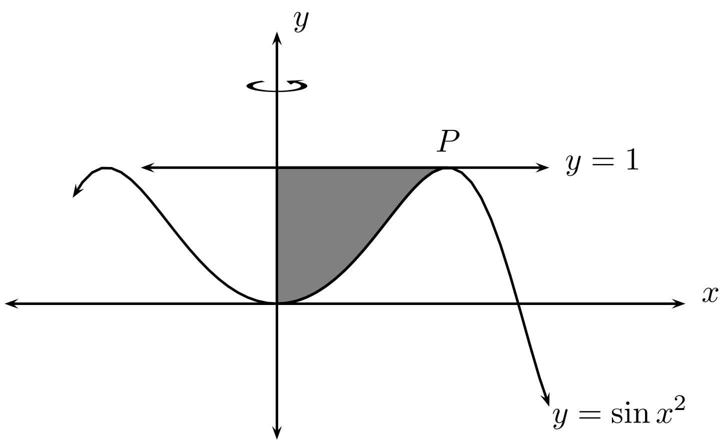
(a) Consider the curve $y = e^{-x}(1 - x)$.

(i) Find and classify the stationary point. **2**

(ii) Sketch the curve showing the main features. (You do not need to find the coordinates of the point of inflexion.) **3**

(iii) Suppose that the line $y = mx$ is a tangent to the curve in the fourth quadrant. Use your diagram to explain why m must be less than $-\dfrac{1}{2e^2}$. **2**

(b)



The region bounded by the curve $y = \sin x^2$, the y-axis and the line $y = 1$ is shaded in the diagram above.

(i) Find the x-coordinate of the point P. **1**

(ii) The shaded region is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the solid generated. Leave your answer in terms of π. **3**

(c) Consider the polynomial $P(z) = z^3 + az^2 + bz + c$ where a, b and c are real. Suppose that ki is a zero of $P(z)$, where k is real and non-zero.

(i) Show that $P(z)$ has one real zero. **2**

(ii) Show that $c = ab$. **2**

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

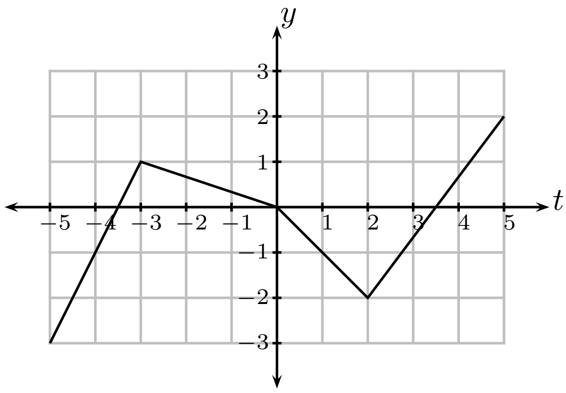
(a) The base of a solid S is the ellipse $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right-angled triangles with hypotenuse in the base. Find the volume of S. **4**

(b) (i) Show that $x^2 \left(1 + x^2\right)^{n-1} = \left(1 + x^2\right)^n - \left(1 + x^2\right)^{n-1}$. **1**

(ii) Suppose that $I_n = \displaystyle\int_0^1 \left(1 + x^2\right)^n \, dx$, where n is a positive integer. **4**

Show that $I_n = \dfrac{1}{2n+1}\left(2^n + 2n\, I_{n-1}\right)$.

(c) **2**



The diagram above shows the graph of the function $y = f(t)$.

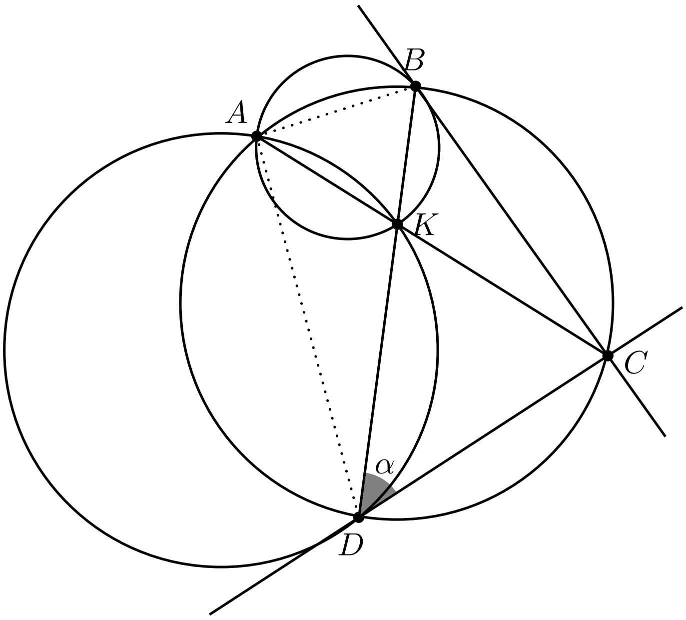
Let $F(x) = \displaystyle\int_{-5}^{x} f(t)\, dt$.

For which value of x does $F(x)$ achieve its absolute minimum value? Give reasons for your answer.

Question Five Continues Over the Page

QUESTION FIVE (Continued)

(d)



In the diagram above, $ABCD$ is a cyclic quadrilateral and diagonals AC and BD intersect at K. Circles AKD and AKB are drawn and it is known that CD is a tangent to circle AKD. Let $\angle BDC = \alpha$.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

(i) Prove that $\triangle BCD$ is isosceles. **2**

(ii) Prove that CB is a tangent to circle AKB. **2**

Exam continues next page ...

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a) A particle of unit mass is projected vertically upwards against a constant gravitational force g and a resistance of magnitude $\dfrac{v}{10}$, where v is the velocity of the particle after t seconds. The initial velocity is 80 metres per second. Take g to be $10\,\text{m/s}^2$.

(i) Taking the positive direction of motion upwards, show that the equation of motion is $\ddot{x} = -\dfrac{v+100}{10}$. **1**

(ii) Show that the time T for the particle to reach its greatest height is given by $T = 10\ln 1{\cdot}8$ seconds. **3**

(iii) Show that the maximum height H attained by the particle is approximately 212 metres. **3**

(iv) From its maximum height the particle falls to its original position under gravity and under the same resistance. **4**
Determine whether the speed at which the particle returns to its starting point is greater than or less than the speed of projection. (Take the positive direction of motion downwards in this part.)

(b) (i) Suppose that x is a positive real number and that n is a positive integer. **1**

Show that $\dfrac{1}{1+x^n} < 1$.

(ii) Let $I_n = \displaystyle\int_0^1 \frac{1}{1+x^n}\,dx$ where n is a positive integer and $n \geq 2$. **3**

Show that $\dfrac{\pi}{4} \leq I_n < 1$.

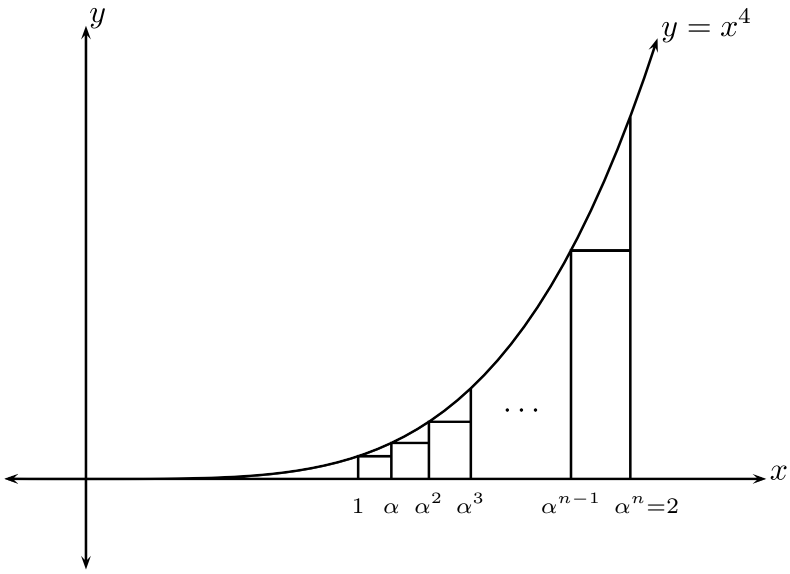
QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that the normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \dfrac{c}{t}\right)$ has equation $ty - t^3x = c(1 - t^4)$. **2**

(ii) Show that from a point $(0, k)$ on the y-axis, exactly two normals can be drawn to the hyperbola. **4**

(iii) Show that there can never be more than four normals drawn to the hyperbola from an arbitrary point in the plane. **1**

(b)



In the diagram above the curve $y = x^4$ is graphed. The interval from $x = 1$ to $x = 2$ is divided into n unequal subintervals. The first subinterval is $1 \le x \le \alpha$, the second is $\alpha \le x \le \alpha^2$, and so on, where $\alpha = 2^{\frac{1}{n}}$.

A rectangle is constructed on each subinterval, as shown in the diagram.

(i) Write down the value of $\lim\limits_{n\to\infty} \alpha$. **1**

(ii) Show that the sum S_n of the areas of the n rectangles is given by **4**

$$S_n = \frac{\alpha^{5n} - 1}{1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4}\ .$$

(iii) Hence evaluate $\lim\limits_{n\to\infty} S_n$. **2**

(iv) What is a simple geometrical interpretation of your answer to part (iii)? **1**

Exam continues next page ...

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $(\alpha^2 - 1)(\alpha^4 - 14\alpha^2 + 1) = \alpha^6 - 15\alpha^4 + 15\alpha^2 - 1$. **1**

(ii) Use de Moivre's theorem to show that $\cot 6\theta = \dfrac{(\alpha^2 - 1)(\alpha^4 - 14\alpha^2 + 1)}{2\alpha(3\alpha^4 - 10\alpha^2 + 3)}$, **3**

where $\alpha = \cot\theta$ (and 6θ is not a multiple of π).

(iii) Hence show that $\cot^2 \dfrac{\pi}{12} + \cot^2 \dfrac{5\pi}{12} = 14$. **2**

(iv) Deduce that $\cot \dfrac{\pi}{12} + \tan \dfrac{\pi}{12} = 4$. **2**

(b) It can be shown that $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$, for $-1 < x < 1$.

(i) Assuming this result, show that $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots\right)$, **2**

where $-1 < x < 1$.

(ii) Hence show that $\ln 2 = 2\left(\dfrac{1}{3} + \dfrac{1}{3 \times 3^3} + \dfrac{1}{5 \times 3^5} + \dfrac{1}{7 \times 3^7} + \cdots\right)$. **2**

(iii) Consider the approximation $\ln 2 \doteqdot 2\left(\dfrac{1}{3} + \dfrac{1}{3 \times 3^3} + \dfrac{1}{5 \times 3^5}\right)$. **3**

Deduce that the error in this approximation is less than $\dfrac{1}{7 \times 2^2 \times 3^5}$.

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

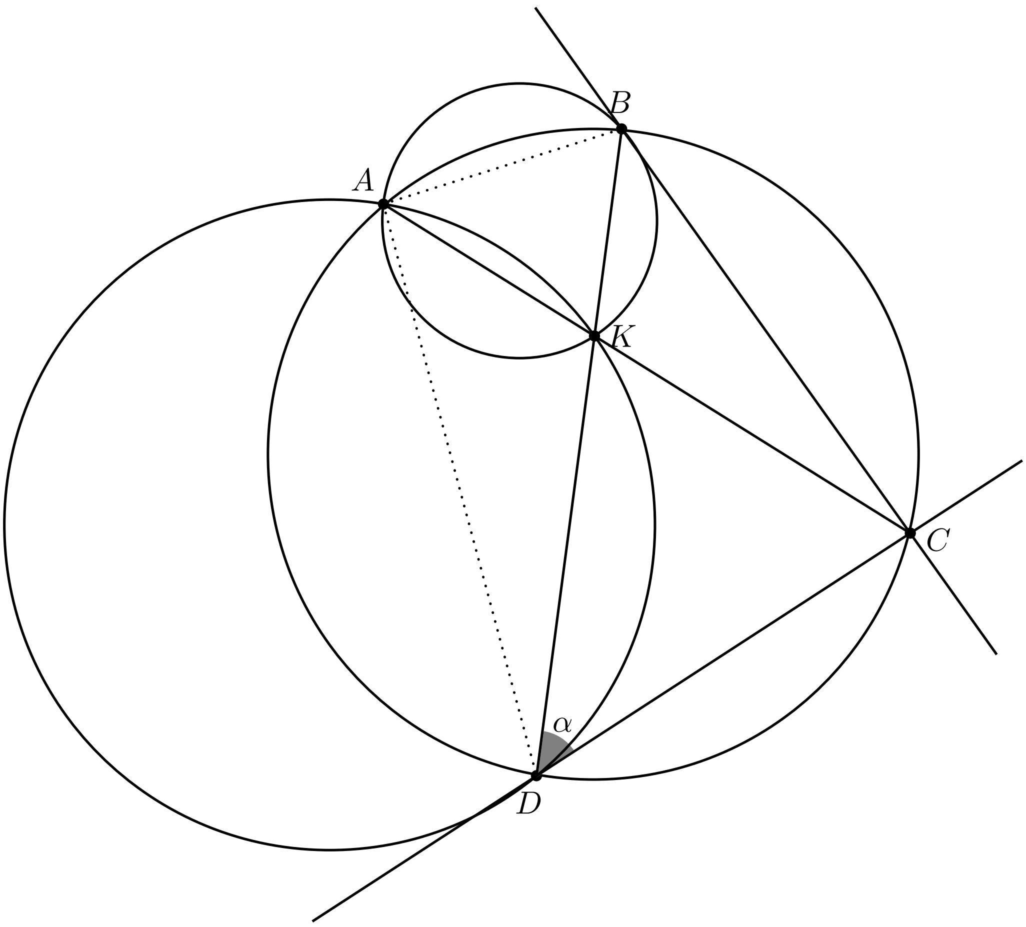
$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

Candidate number: ....

Detach this sheet and bundle it with the rest of question five.

QUESTION FIVE

(d)



Sydney Grammar Ext 2 2010 Solutions

Question 1

a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int e^{u}\,du$

$= 2e^{u} + c$

$= 2e^{\sqrt{x}} + c$

b) (i) $A + B(x-1) = 2x - 1$

$\underline{x = 1}$ $A = 1$

equate coefficient of x

$B = 2$

$\therefore A = 1,\ B = 2$

(ii) $\displaystyle\int \frac{2x-1}{(x-1)^2}\,dx = \int\left[\frac{1}{(x-1)^2} + \frac{2}{x-1}\right]dx$

$\displaystyle = -\frac{1}{x-1} + 2\log(x-1) + c$

c) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx = \int \frac{dx}{\sqrt{4 - (x-1)^2}}$

$\displaystyle = \sin^{-1}\left(\frac{x-1}{2}\right) + c$

d) $\displaystyle\int x\sec^2 x\,dx$

$u = x$ $\quad v = \tan x$

$du = dx$ $\quad dv = \sec^2 x\,dx$

$\displaystyle = x\tan x - \int \tan x\,dx$

$= x\tan x + \log\cos x + c$

e) $\displaystyle\int_0^2 \frac{1}{(4+x^2)^2}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$x = 0,\ \theta = 0$

$x = 2,\ \theta = \frac{\pi}{4}$

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta\,d\theta}{(4+4\tan^2\theta)^2}$

$\displaystyle = \frac{1}{8}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$

$\displaystyle = \frac{1}{8}\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$

$\displaystyle = \frac{1}{16}\int_0^{\frac{\pi}{4}} (1 + \cos 2\theta)\,d\theta$

$\displaystyle = \frac{1}{16}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$\displaystyle = \frac{1}{16}\left(\frac{\pi}{4} + \frac{1}{2} - 0\right) = \frac{\pi + 2}{64}$

Question 2

a)(i) $wz = (2-i)(3+i)$
$= 6+2i-3i+1$
$= \underline{\underline{7-i}}$

(ii) $\overline{w}\,\overline{z} = \overline{wz}$
$= \underline{\underline{7+i}}$

(iii) $\frac{w}{z} = \frac{2-i}{3+i} \times \frac{3-i}{3-i}$
$= \frac{6-2i-3i-1}{9+1}$
$= \frac{5-5i}{10}$
$= \underline{\underline{\frac{1-i}{2}}}$

b) (i) $|1-i\sqrt{3}|$
$= \sqrt{1^2+(\sqrt{3})^2}$
$= 2$

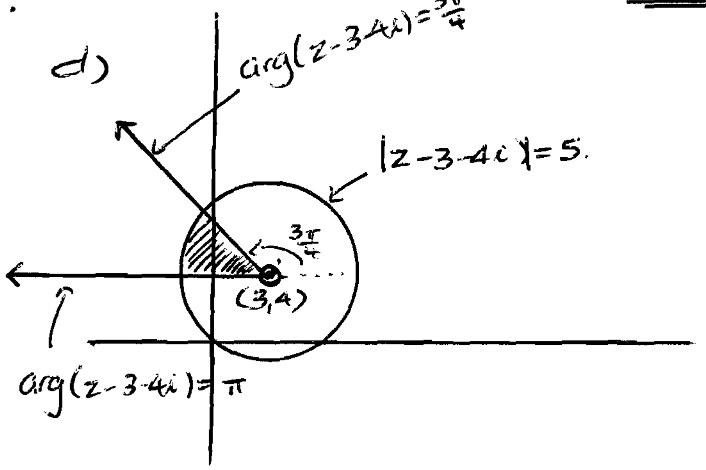
$\arg(1-i\sqrt{3}) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$
$= -\frac{\pi}{3}$

$\therefore\ 1-i\sqrt{3} = \underline{\underline{2\,\text{cis}\left(-\frac{\pi}{3}\right)}}$

(ii) $(1-i\sqrt{3})^9 = \left[2\,\text{cis}\left(\frac{-\pi}{3}\right)\right]^9$
$= 2^9\,\text{cis}(-3\pi)$
$= -2^9 = \underline{\underline{-512}}$

c) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} = \frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} \times \frac{\cos 2\theta + i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$
$= \frac{\cos 5\theta + i\sin 5\theta}{\cos^2 2\theta + \sin^2 2\theta}$
$= \underline{\underline{\text{cis}\,5\theta}}$

d)



e) (i) $Re\left(z - \frac{1}{\bar{z}}\right) = 0$

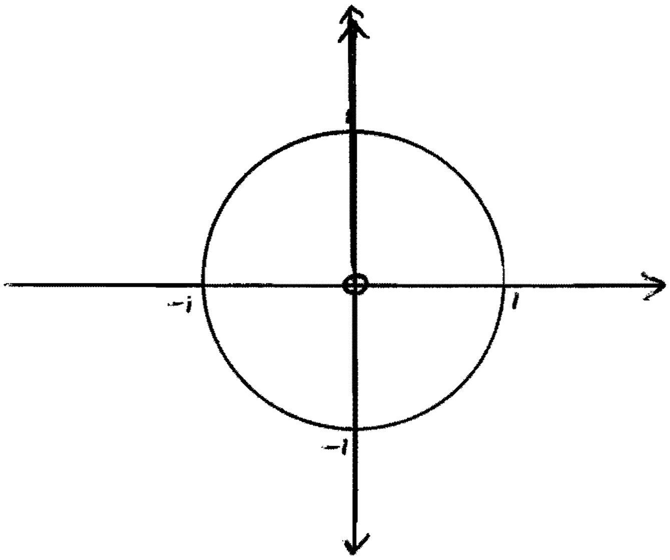
$$Re\left(z - \frac{z}{|z|^2}\right) = 0$$

$$Re\left(\frac{(x+iy)(x^2+y^2) - x - iy}{x^2+y^2}\right) = 0$$

$$x^3 + xy^2 - x = 0$$

$$\underline{\underline{x(x^2+y^2-1) = 0}}$$

(ii)



locus is
circle $x^2+y^2=1$
and
line $x=0$ (ie y axis)
excluding $(0,0)$
(as $\bar{z} \neq 0$.)

Question 3

a) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

(i) $b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$e^2 - 1 = -\dfrac{9}{25}$

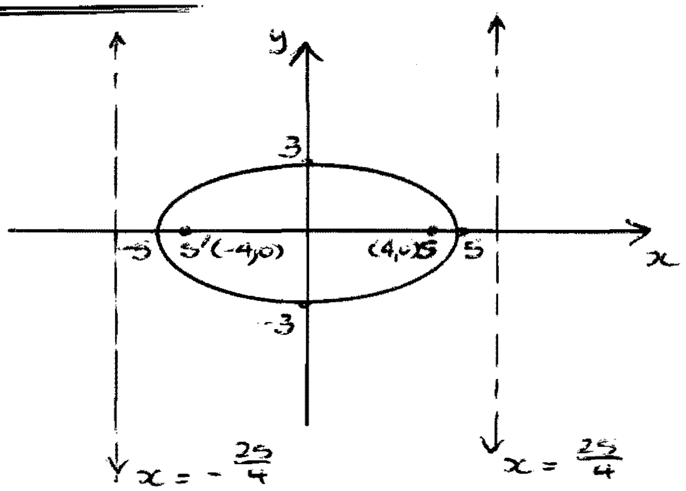
$e^2 = \dfrac{16}{25}$

$\underline{e = \dfrac{4}{5}}$

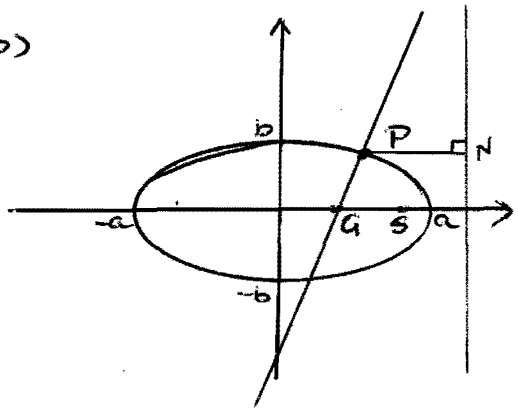
(ii) foci: $(\pm 4, 0)$

directrices: $x = \pm \dfrac{25}{4}$

(iii)



b)



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{2x}{a^2} + \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-b^2x}{a^2y}$

at P, $\dfrac{dy}{dx} = \dfrac{-b^2x_1}{a^2y_1}$

$\therefore$ slope of normal is $\dfrac{a^2y_1}{b^2x_1}$

(ii) $m_{PG} = \dfrac{a^2y_1}{b^2x_1}$

$\dfrac{y_1}{x_1 - x} = \dfrac{a^2y_1}{b^2x_1}$

$a^2(x_1 - x) = b^2x_1$

$a^2x = (a^2 - b^2)x_1$

$x = \dfrac{a^2-b^2}{a^2}x_1$

now $b^2 = a^2(\cancel{d}1 - e^2)$

$x = \dfrac{a^2 - a^2(1-e^2)}{a^2}x_1$

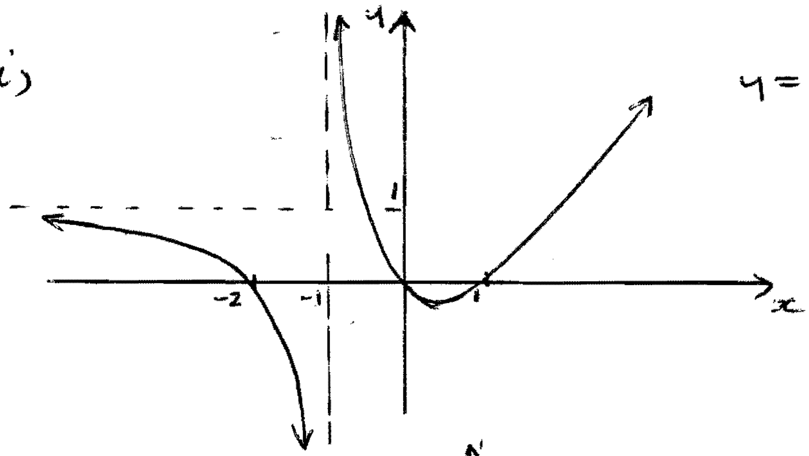
$= \underline{e^2x_1}$

(iii) $SG = ae - e^2 x_1$
$= e(a - ex_1)$

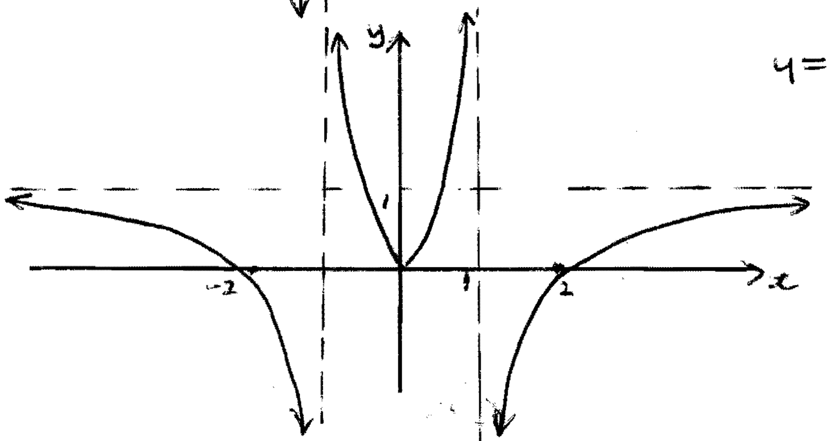
$PS = e\,PN$
$= e\left(\frac{a}{e} - x_1\right)$
$= a - ex_1$

$\therefore \underline{\underline{SG = e\,PS}}$

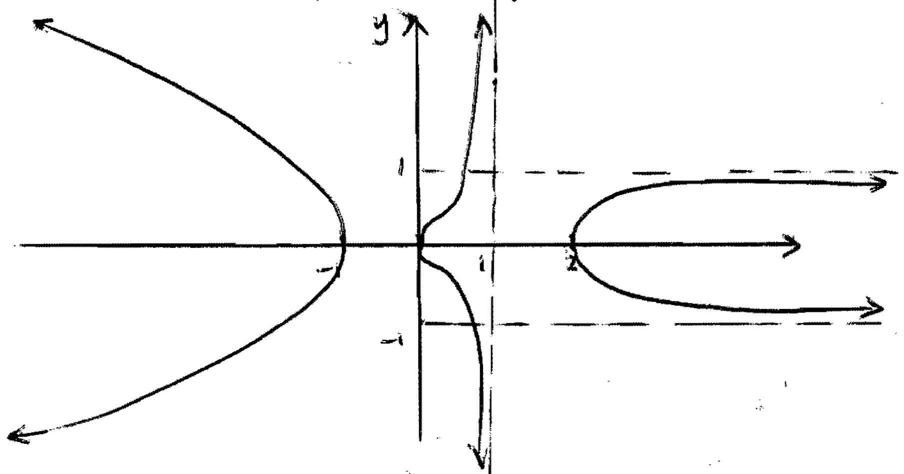
c) (i) $y = f(-x)$



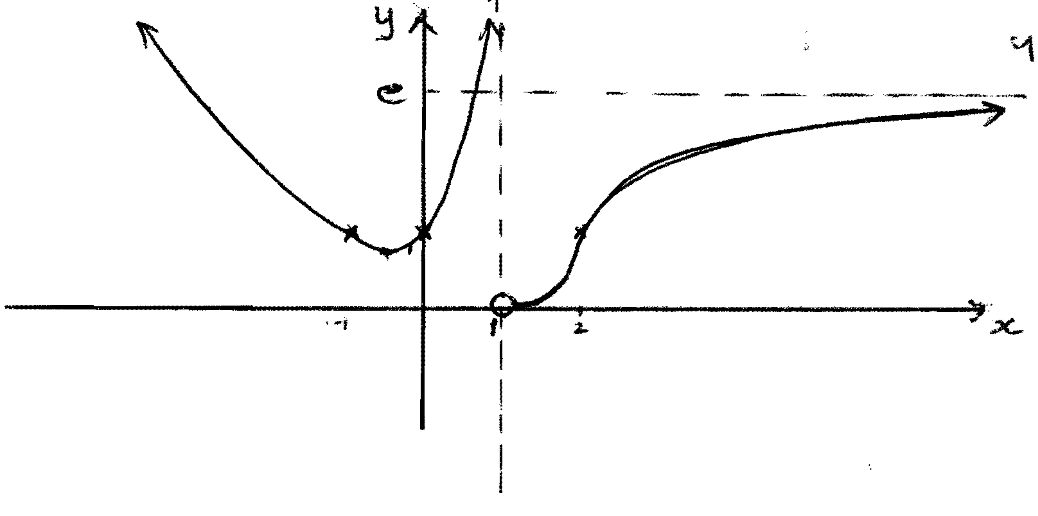
(ii) $y = f(|x|)$



(iii) $y^2 = f(x)$



(iv) $y = e^{f(x)}$



Question 4

a) $y = e^{-x}(1-x)$

$y' = e^{-x}(-1) + (1-x)(-e^{-x})$

$= e^{-x}(x-2)$

stationary pts occur when $y'=0$

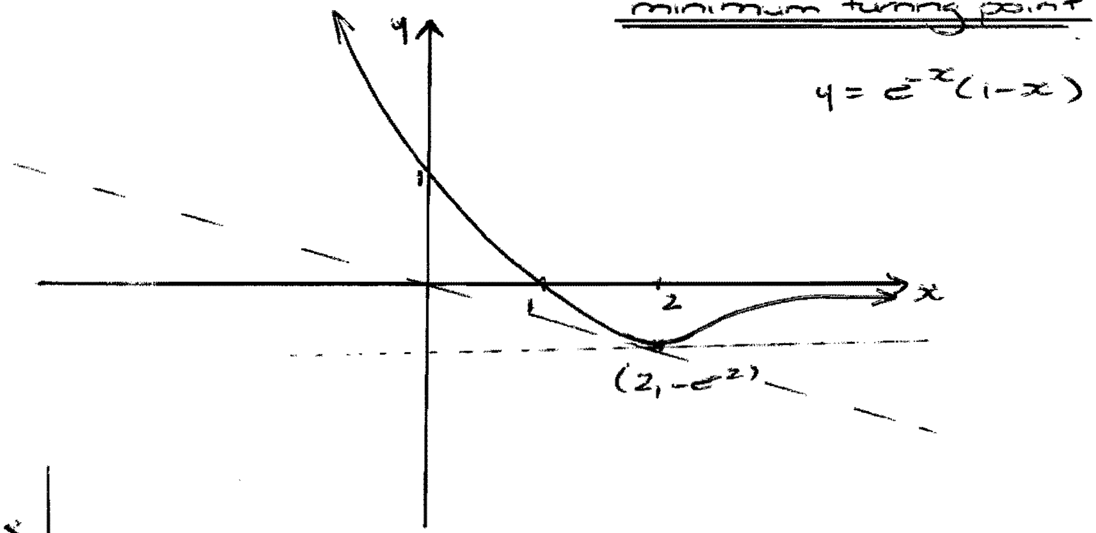
i.e. $e^{-x}(x-2)=0$

$x=2$

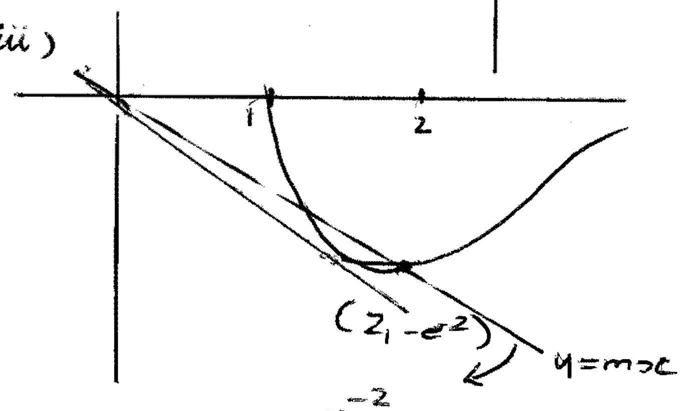
x	1	2	3
y'	<0	0	>0

$\therefore (2, -e^{-2})$ is a minimum turning point

(ii)



(iii)



$m_{OS} = \frac{-e^{-2}}{2}$

$= -\frac{1}{2e^2}$

The line joining the origin to the stationary point cuts the curve twice.

Thus to become a tangent the line must rotate clockwise

i.e. $m_T < m_{OS}$,

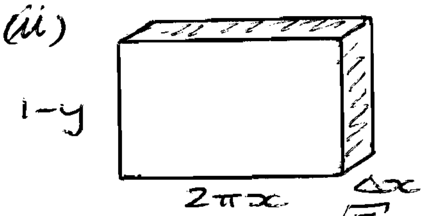
$m_T < -\frac{1}{2e^2}$

b) (i) $\sin x^2 = 1$

$$x^2 = \frac{\pi}{2}$$

$$\underline{\underline{x = \sqrt{\frac{\pi}{2}}}}$$

(ii)



$$A(x) = 2\pi x(1-y)$$

$$= 2\pi x(1-\sin x^2)$$

$$\Delta V = 2\pi x(1-\sin x^2)\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{\sqrt{\frac{\pi}{2}}} 2\pi x(1-\sin x^2)\Delta x$$

$$= 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} (x - x\sin x^2)\,dx$$

$$= 2\pi \left[\frac{x^2}{2} + \cos x^2 \right]_0^{\sqrt{\frac{\pi}{2}}}$$

$$= 2\pi \left(\frac{\pi}{4} + 0 - 0 - 1 \right)$$

$$\underline{\underline{= \left(\frac{\pi^2}{2} - 2\pi \right) \text{units}^3}}$$

c) (i) $P(z) = z^3 + az^2 + bz + c$ has three zeroes.

If ki is a zero, so is $-ki$ as coefficients are real and must appear in conjugate pairs.

Thus there is only one other zero, which must be real as its conjugate pair cannot exist.

$\therefore$ $\underline{\underline{P(z) \text{ has one real zero.}}}$

(ii)

$$\alpha + ki - ki = -a$$
$$\alpha = -a$$

$$(\alpha)(ki) + (\alpha)(-ki) + (ki)(-ki) = b$$
$$k^2 = b$$

$$(\alpha)(ki)(-ki) = -c$$

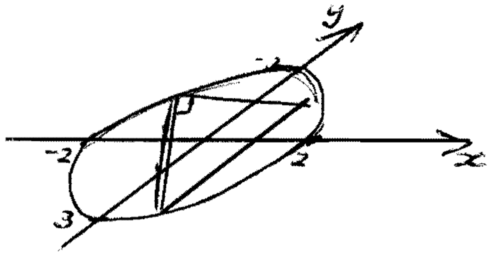
$$-ak^2 = -c$$

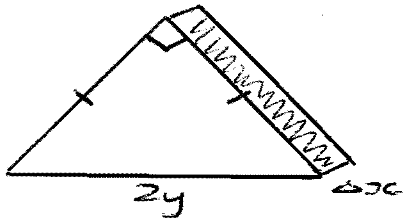
$$-ab = -c$$

$$\underline{\underline{c = ab}}$$

Question 5

a)





$$A(x) = \frac{1}{4} \times 2y \times 2y$$
$$= y^2$$
$$= 9 - \frac{9}{4}x^2$$

$$\Delta V = (9 - \frac{9}{4}x^2)\Delta x$$
$$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} (9 - \frac{9}{4}x^2)\Delta x$$
$$= 18\int_0^2 (1 - \frac{1}{4}x^2)\,dx$$
$$= 18\left[x - \frac{1}{12}x^3\right]_0^2$$
$$= 18(2 - \frac{2}{3} - 0)$$
$$= \underline{\underline{24 \text{ units}^3}}$$

b)(i) $x^2(1+x^2)^{n-1} = (1+x^2-1)(1+x^2)^{n-1}$
$$= (1+x^2)(1+x^2)^{n-1} - (1+x^2)^{n-1}$$
$$= \underline{\underline{(1+x^2)^n - (1+x^2)^{n-1}}}$$

(ii) $I_n = \int_0^1 (1+x^2)^n\,dx$ $\qquad u = (1+x^2)^n \qquad v = x$

$$= \left[x(1+x^2)^n\right]_0^1 - 2n\int_0^1 x^2(1+x^2)^{n-1}\,dx \qquad du = n(1+x^2)^{n-1}2x\,dx \qquad dv = dx$$
$$= 2^n - 2n\left\{\int_0^1 (1+x^2)^n\,dx - \int_0^1 (1+x^2)^{n-1}\,dx\right\}$$
$$= 2^n - 2nI_n + 2nI_{n-1}$$
$$(2n+1)I_n = 2^n + 2nI_{n-1}$$
$$\underline{\underline{I_n = \frac{1}{2n+1}\left(2^n + 2nI_{n-1}\right)}}$$

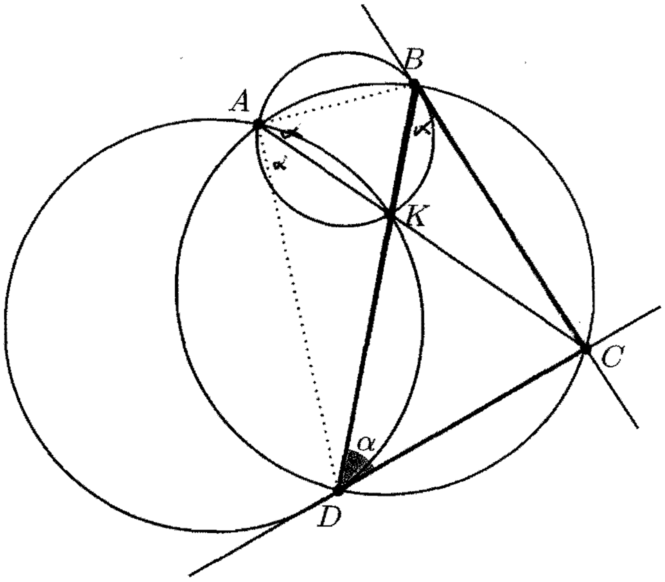
c) $F(-3.5) = -\frac{1}{2} \times 1.5 \times 3$
$= -2.25$

$F(0) = -2.25 + \frac{1}{2} \times 3.5 \times 1$
$= -0.5$

$F(3.5) = -0.5 - \frac{1}{2} \times 3.5 \times 2$
$= -4$

$\therefore$ when $\underline{x = 3.5}$, $F(x)$ achieves its absolute minimum, after this point $y = f(t)$ is ≥ 0 and will thus increase the value of $F(x)$.

d)



(i) $\angle KDC = \angle DAK = \alpha$ (alternate segment theorem)

$\angle DAK = \angle DBC = \alpha$ ($\angle$'s in same segment $=$)

$\therefore \angle KDC = \angle DBC = \alpha$

$\underline{\triangle BCD \text{ is isosceles}}$ ($2 = \angle$'s)

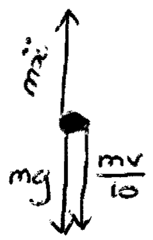
(ii) $\angle CDB = \angle CAB = \alpha$ ($\angle$'s in same segment $=$)

$\therefore \angle CBD = \angle BAC$

Thus BD is a tangent to circle at B as $\angle$ in alternate segment $=$ $\angle$ between tangent and chord.

Question 6

a)



$$m\ddot{x} = -mg - \frac{mv}{10}$$

$$m = 1,\ g = 10$$

$$\ddot{x} = -10 - \frac{v}{10}$$

$$= -\frac{v+100}{10}$$

(ii)

$$\frac{dv}{dt} = -\frac{v+100}{10}$$

$$\int_0^T dt = -\int_{80}^{0} \frac{10\,dv}{v+100}$$

$$T = 10\left[\log(v+100)\right]_0^{80}$$

$$= 10\log\left(\frac{180}{100}\right)$$

$$= \underline{10\log 1.8} \text{ seconds}$$

(iii)

$$v\frac{dv}{dx} = -\frac{v+100}{10}$$

$$\int_0^H dx = -\int_{80}^{0} \frac{10v\,dv}{v+100}$$

$$H = 10\int_0^{80}\left[1 - \frac{100}{v+100}\right]dv$$

$$= 10\left[v - 100\log(v+100)\right]_0^{80}$$

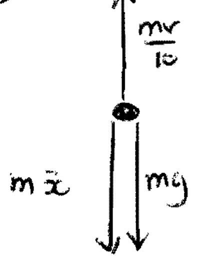
$$= 10\left(80 - 100\log\left(\frac{180}{100}\right)\right)$$

$$= 100(8 - 10\log 1.8)$$

$$= 212.213335\,1$$

$$= \underline{212\ m} \quad (\text{to nearest metre})$$

(iv)



$$m\ddot{x} = mg - \frac{mv}{10}$$

$$m = 1,\ g = 10$$

$$\ddot{x} = 10 - \frac{v}{10}$$

$$= \frac{100-v}{10}$$

$$v\frac{dv}{dx} = \frac{100-v}{10}$$

$$\int_0^H dx = \int_0^V \frac{10v\,dv}{100-v}$$

$$H = 10\int_0^V \left[-1 + \frac{100}{100-v}\right] dv$$

$$= 10\left[-v - 100\log(100-v)\right]_0^V$$

$$= 10\left(-V - 100\log\left(\frac{100-V}{100}\right)\right)$$

when $V = 80$

$$H = 10\left(-80 - 100\log\frac{20}{100}\right)$$

$$= 809.4379124$$

$\therefore$ when $H = 212m$, V would be <u>less then the speed of projection.</u>

b) (i) If $x > 0$ then $x^n > 0$

$$1 + x^n > 1$$

$$\frac{1}{1+x^n} < 1$$

(ii)

$$\int_0^1 \frac{dx}{1+x^2} \leq I_n < \int_0^1 dx$$

$$\left[\tan^{-1}x\right]_0^1 \leq I_n < \left[x\right]_0^1$$

$$\frac{\pi}{4} - 0 \leq I_n < 1 - 0$$

$$\frac{\pi}{4} \leq I_n < 1$$

Question 7

a) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

when $x = ct$, $\frac{dy}{dx} = \frac{-c^2}{c^2 t^2}$

$= -\frac{1}{t^2}$

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3(x - ct)$

$= t^3 x - ct^4$

$ty - t^3 x = c - ct^4$

$\underline{ty - t^3 x = c(1 - t^4)}$

(ii) $(0, k)$:

$tk = c(1 - t^4)$

$ct^4 + kt - c = 0$

$\Sigma \alpha^2 = (\Sigma \alpha)^2 - 2\Sigma \alpha\beta$

$= 0^2 - 2(0)$

$= 0$

As the sum of (roots squared) = 0, imaginary roots must exist as it is impossible to add four real squares to total zero.

Imaginary roots occur in conjugate pairs

∴ either zero or two real roots.

We know that there exists normals to the hyperbola

∴ there are exactly two solutions to $ct^4 + kt - c = 0$

ie <u>exactly two normals can be drawn to the hyperbola</u>

(iii) (a, b): $tb - t^3 a = c(1 - t^4)$

$ct^4 - at^3 + bt - c = 0$

A quartic has a maximum of four real roots

∴ <u>there can never be more than four normals drawn to the hyperbola from an arbitrary point.</u>

b) (i) $\lim_{n\to\infty} \alpha = \lim_{n\to\infty} 2^{\frac{1}{n}}$

$= 2^0$

$= \underline{\underline{1}}$

(ii) $S_n = 1^4(\alpha-1) + \alpha^4(\alpha^2-\alpha) + (\alpha^2)^4(\alpha^3-\alpha^2) + (\alpha^3)^4(\alpha^4-\alpha^3) + \dots$

$= (\alpha-1) + \alpha^5(\alpha-1) + \alpha^{10}(\alpha-1) + \alpha^{15}(\alpha-1) + \dots$

GP: $a = (\alpha-1)$, $r = \alpha^5$

$S_n = \dfrac{(\alpha-1)((\alpha^5)^n-1)}{\alpha^5-1}$

$= \dfrac{(\alpha-1)(\alpha^{5n}-1)}{(\alpha-1)(\alpha^4+\alpha^3+\alpha^2+\alpha+1)}$

$= \underline{\underline{\dfrac{\alpha^{5n}-1}{1+\alpha+\alpha^2+\alpha^3+\alpha^4}}}$

(iii) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{(2^{\frac{1}{n}})^{5n}-1}{1+\alpha+\alpha^2+\alpha^3+\alpha^4}$

$= \lim_{n\to\infty} \dfrac{2^5-1}{1+\alpha+\alpha^2+\alpha^3+\alpha^4}$

$= \dfrac{2^5-1}{5}$

$= \underline{\underline{\dfrac{31}{5}}}$

(iv) <u>The area under the curve from $x=1$ to $x=2$</u>

<u>is $\dfrac{31}{5}$</u>

Question 8

a) $(\alpha^2-1)(\alpha^4-14\alpha^2+1) = \alpha^6-14\alpha^4+\alpha^2-\alpha^4+14\alpha^2-1$

$= \underline{\underline{\alpha^6-15\alpha^4+15\alpha^2-1}}$

$(\cos\theta+i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$= c^6+6ic^5s-15c^4s^2-20ic^3s^3+15c^2s^4+6ics^5-s^6$

$\cot 6\theta = \dfrac{\cos 6\theta}{\sin 6\theta}$

$= \dfrac{\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta}{6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta} \div \dfrac{\sin^6\theta}{\sin^6\theta}$

$= \dfrac{\cot^6\theta-15\cot^4\theta+15\cot^2\theta-1}{6\cot^5\theta-20\cot^3\theta+6\cot\theta}$

$= \dfrac{\alpha^6-15\alpha^4+15\alpha^2-1}{6\alpha^5-20\alpha^3+6\alpha}$

$= \underline{\underline{\dfrac{(\alpha^2-1)(\alpha^4-14\alpha^2+1)}{2\alpha(3\alpha^4-10\alpha^2+3)}}}$

(ii) $\cot 6\theta = 0$

$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

$\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}$

Now $\cot 6\theta = 0$

$\dfrac{(\alpha^2-1)(\alpha^4-14\alpha^2+1)}{2\alpha(3\alpha^4-10\alpha^2+3)} = 0$, where $\alpha = \cot\theta$

$(\alpha^2-1)(\alpha^4-14\alpha^2+1) = 0$

$\alpha^2 = 1$ or $\alpha^4-14\alpha^2+1=0$

$\alpha = \pm 1$

$\cot\theta = \pm 1$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore \alpha^4-14\alpha^2+1=0$

has roots $\cot\frac{\pi}{12}, \cot\frac{5\pi}{12}, \cot\frac{7\pi}{12}, \cot\frac{11\pi}{12}$

$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$= 0^2 - 2(-14)$

$= 28$

$$\cot^2\frac{\pi}{12} + \cot^2\frac{5\pi}{12} + \cot^2\frac{7\pi}{12} + \cot^2\frac{11\pi}{12} = 28$$

$$\cot\frac{\pi}{12} = -\cot\frac{11\pi}{12}, \quad \cot\frac{5\pi}{12} = -\cot\frac{7\pi}{12}$$

$$\therefore \cot^2\frac{\pi}{12} + \cot^2\frac{5\pi}{12} + \cot^2\frac{5\pi}{12} + \cot^2\frac{11\pi}{12} = 28$$

$$2\cot^2\frac{\pi}{12} + 2\cot^2\frac{5\pi}{12} = 28$$

$$\underline{\underline{\cot^2\frac{\pi}{12} + \cot^2\frac{5\pi}{12} = 14}}$$

(iv) $\cot\frac{5\pi}{12} = \tan\frac{\pi}{12}$ (complementary ratios)

$$\cot^2\frac{\pi}{12} + \tan^2\frac{\pi}{12} = 14$$

$$\left(\cot\frac{\pi}{12} + \tan\frac{\pi}{12}\right)^2 - 2\cot\frac{\pi}{12}\tan\frac{\pi}{12} = 14$$

$$\left(\cot\frac{\pi}{12} + \tan\frac{\pi}{12}\right)^2 - 2 = 14$$

$$\left(\cot\frac{\pi}{12} + \tan\frac{\pi}{12}\right)^2 = 16$$

$$\underline{\underline{\cot\frac{\pi}{12} + \tan\frac{\pi}{12} = 4}} \qquad \left(\cot\frac{\pi}{12} > 0,\ \tan\frac{\pi}{12} > 0\right)$$

b) $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right]$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \ldots$$

$$= \underline{\underline{2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)}}$$

(ii) $\ln 2 = \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$

$$= 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots\right)$$

$$= \underline{\underline{2\left(\frac{1}{3} + \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} + \frac{1}{7\times 3^7} + \ldots\right)}}$$

(iii)

$$\text{error} = 2\left(\frac{1}{7\times 3^7} + \frac{1}{9\times 3^9} + \frac{1}{11\times 3^{11}} + \cdots\right)$$

$$< 2\left(\frac{1}{7\times 3^7} + \frac{1}{7\times 3^9} + \frac{1}{7\times 3^{11}} + \cdots\right)$$

$$= \frac{2}{7}\left(\frac{1}{3^7} + \frac{1}{3^9} + \frac{1}{3^{11}} + \cdots\right)$$

$$= \frac{2}{7}\left(\frac{\frac{1}{3^7}}{1-\frac{1}{3^2}}\right)$$

$$= \frac{2}{7}\left(\frac{1}{3^7 - 3^5}\right)$$

$$= \frac{2}{7}\times\frac{1}{3^5(3^2-1)}$$

$$= \frac{2}{7}\times\frac{1}{3^5\times 8}$$

$$= \frac{1}{7\times 3^5\times 4}$$

$$= \frac{1}{7\times 3^5\times 2^2}$$

$$\therefore \text{error} < \frac{1}{7\times 3^5\times 2^2}.$$